2104529 Computational Method for IE
Workshop 2: Mathematics in R

Question 1

$$\mathbf{A} = \begin{bmatrix} 7 & 1 \\ 4 & -3 \\ 2 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 7 \\ 0 & -1 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ -1 & 4 & 7 \end{bmatrix}$$

(a) add row vector $\mathbf{b_3} = [3, 8, 0]$ to matrix $\mathbf{B}$, create matrix $\acute{\mathbf{B}} \equiv \begin{bmatrix} \mathbf{B} & \mathbf{I} \end{bmatrix}$

(b) access the first two rows and columns of matrix $\mathbf{C}$

(c) find relationship among $(\mathbf{AB})\mathbf{C}$, $\mathbf{A}(\mathbf{BC})$, $\mathbf{C^T B^T A^T}$ and $(\mathbf{ABC})^\mathbf{T}$

(d) find relationship among $\det(\mathbf{C})$, $\det(\mathbf{C}^3)$, $\det(\mathbf{C})^3$, $\det(\mathbf{C}^{-1})$, and $\det(10\,\mathbf{C})$

(e) find eigenvalue and eigenvector of $\mathbf{C}$ and $\mathbf{C^T}$

(f) [0 points (bonus)] find triangle decomposition of $\mathbf{C}$ or ($\mathbf{C} \equiv \mathbf{S^{-1} \Lambda S}$) and show that $\mathbf{C}^2 = \mathbf{S^{-1}}\mathbf{\Lambda}^2\mathbf{S}$

Question 2

consider the following matrix $\mathbf{E}$

$$\mathbf{E} \quad = \quad \begin{bmatrix} 0.50 & 0.87 & 1.00 & 0.87 & 0.50 & 0.00 & -0.50 \\ 0.87 & 0.50 & 0.00 & -0.50 & -0.87 & -1.00 & -0.87 \end{bmatrix}$$

Visualize and explain the following matrix multiplication as the geometric process of points in $\mathbf{E}$

$$(a) \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix} \times \mathbf{E} \quad , \quad (b) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \mathbf{E} \quad , \quad (c) \begin{bmatrix} \sin(\frac{\pi}{6}) & -\cos(\frac{\pi}{6}) \\ \cos(\frac{\pi}{6}) & \sin(\frac{\pi}{6}) \end{bmatrix} \times \mathbf{E}$$

$$(d) \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \times \mathbf{E} \quad , \quad (e) \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \times \mathbf{E} \quad , \quad (f) \begin{bmatrix} 1 & 0 \\ \sin(\frac{\pi}{6}) & -\cos(\frac{\pi}{6}) \\ \cos(\frac{\pi}{6}) & \sin(\frac{\pi}{6}) \end{bmatrix} \times \mathbf{E}$$

Question 3

$$\mathbf{Ax} = \mathbf{b} \qquad \text{or} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} \qquad \text{note } \acute{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \text{ and } \acute{\mathbf{x}}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

(a) Show that determinant of $\mathbf{A}$ is 0 (**hint:** what the result imply?)

(b) [0 points (bonus)] What are the ranks of matrix $\mathbf{A}$ and augment matrix $\mathbf{A}|\mathbf{b}$?

(c) Show that the following linear system has an infinite number of solutions (**hints:** any affine combination of $\acute{\mathbf{x}}_1$ and $\acute{\mathbf{x}}_2$ is a solution)

(d) solve $\mathbf{x}$ for system of linear equation $\mathbf{A}'|\mathbf{b}'$ (**hints:** how to compute $\acute{\mathbf{x}}_1$ and $\acute{\mathbf{x}}_2$)

Question 4

Plot the level set of the following functions (**hint:** use R command contour $(\cdot)$)

(a) $f(x,y) = (x+y)\, e^{-(x+y)}$

(b) $g(x_1,x_2) = 100(x_2 - x_1^2)^2 + (1-x_1)^2$

Question 5

Consider function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{2x^6 - 4x^4 + 9x^2 + 12x - 18}{x^4 + x^2 - 12x - 12}$$

(a) plot function $f(x)$ for $x \in [-100, 100]$ using R command curve $(\cdot)$

(b) find the optimal solution using First Order Condition (R command uniroot $(\cdot)$)

(c) find the optimal solution using R command optimizie $(\cdot)$

Question 6

Find the closed-form of gradient vector and hessian and compute their values of the following function at given point.

(a) $f_1(x) = x^2 7\, x + 3\, e^x \, \sin(\frac{1}{6}\pi\, x)$ at $x = 3$

(b) $f_2(x_1,x_2) = x_1^2 + x_1 x_2 \, \sin(\frac{1}{2}\pi\, x_1\, x_2) + x_2^2$ at $(x_1,x_2) = (3,5)$

(c) $f_3(x_1,x_2,x_3) = 100(x_3 x_2^2)^2 + (x_2 1)^2 + 10\,(x_2 x_1^2)^2 + (x_1 1)^2$ at $(x_1,x_2,x_3) = (0,0,0)$

Question 7

Find the values of gradient vector and hessian matrix of the following function at given points. (**hint:** no closed form required)

(a) $f_1(x,y) = x^2 + y^3 - 3\,x - 3\,y + 5$ at $(x,y) = (-1,2)$

(b) $f_2(x,y,z) = 100(y - x^2)^2 + (1 - x^2)$ at $(x,y) = (1/2, 4/3)$

(c) $f_3(x,y,z) = \cos(x\, y) - \sin(x\, z)$ at $(x,y,z) = (0, \pi, \pi/2)$

note: What are properties of these functions

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